

NASA Contractor Report 3921

NASA-CR-3921 19850025800

Scalar/Vector Potential Formulation for Compressible Viscous Unsteady Flows

Luigi Morino

CONTRACT NAS1-17317
AUGUST 1985

LIBRARY COPY

RECEIVED
LANGLEY RESEARCH CENTER
AUGUST 1985

NASA

NASA Contractor Report 3921

Scalar/Vector Potential Formulation for Compressible Viscous Unsteady Flows

Luigi Morino

*Institute for Computer Application Research
and Utilization in Science, Inc.
Boston, Massachusetts*

Prepared for
Langley Research Center
under Contract NAS1-17317



National Aeronautics
and Space Administration

Scientific and Technical
Information Branch

1985

TABLE OF CONTENTS

	<u>Page</u>
Section 1 - Introduction	1.1
1.1 Scalar/Vector Potential Methods for Incompressible Viscous Flows	1.1
1.2 A Scalar/Vector Potential Method for Incompressible Flows	
Section 2 - Foundations of Continuum Mechanics	2.1
2.1 Basic Definitions	2.1
2.2 Fundamental Conservation Principles	2.2
2.3 Jacobian of Transformation	2.2
2.4 Time Derivative of Integrals	2.3
2.5 Continuity Equation	2.4
2.6 Stress Tensor	2.4
2.7 Cauchy's Equation of Motion	2.5
Figure 2.1 Cauchy tetrahedron	2.6
2.8 Symmetry of Stress Tensor	2.7
2.9 Energy Transfer Equation and Virtual Work Principle	2.7
2.10 Thermodynamic Energy Equation	2.9
2.11 Heat Flux Vector	2.9
2.12 Thermodynamic Energy Equation in Differential Form	2.9
2.A A Convenient Expression for the Acceleration	2.10
2.B Kinematics of Deformation	2.10
Section 3 - Entropy and the Second Law of Thermodynamics	3.1
3.1 Starting Equations	3.1
3.2 Equation of State	3.2
3.3 Entropy Evolution Equation	3.2
3.4 Second Law of Thermodynamics	3.3
3.5 Thermodynamic Pressure and Mechanical Pressure	3.4
3.6 Enthalpy	3.4
3.A Dependence of Internal Energy on Specific Volume	3.5
Section 4 - Vector and Scalar Potentials	4.1
4.1 Starting Equations	4.1
4.2 Decomposition Theorem	4.1
4.3 Generalized Bernoullian Theorem	4.2
4.4 Differential Equation for Potential	4.4
4.5 Vorticity Dynamics Equation	4.5
4.6 Summary of results	4.6
4.A Equation of State for an Ideal Gas	4.8
4.B Constitutive Relations	4.10
4.C Speed of Sound	4.11

Section 5 - Concluding Remarks	5.1
5.1 Comments	5.1
5.2 Recommendation for Future Work	5.2
Section 6 - References	6.1

LIST OF SYMBOLS

\bar{A}	vector potential
B	see Eq. 4.4.5
c_v	specific heat coefficient at constant volume
c_p	specific heat coefficient at constant pressure
\bar{D}	deformation tensor, Eq. 2.B.5
e	specific internal energy
\bar{f}	force per unit mass
f_D	see Eq. 4.3.6
H	see Eq. 4.3.1
\bar{i}_k	Cartesian base vectors
\bar{I}	unit tensor
J	Jacobian of transformation $\bar{x}(\xi^a)$
p	pressure
Q	heat flux
\bar{q}	heat flux vector
r	$ \bar{x} - \bar{x}_s $
R	ideal gas constant
S	entropy
\bar{t}	stress vector
T	temperature
\bar{T}	stress tensor
\bar{v}	velocity of fluid
\bar{v}_B	velocity of body
\bar{v}_v	velocity induced by vortices
V	material volume

V_0	image of V_M in ξ^α -space
\bar{v}	viscous stress tensor
\bar{x}	position vector
$\bar{\zeta}$	vorticity
θ	expansion, $\text{div } \bar{v}$
μ	viscosity coefficient
ξ^α	material curvilinear coordinates
ρ	density
σ	surface of V_M
τ	specific volume
φ	scalar potential
Φ	dissipation function, Eq. 3.3.6
$\bar{\omega}$	angular velocity
Ω	potential energy for \bar{f} , Eq. 4.3.3
$\bar{\Omega}$	rotation tensor

Special Symbols

$\frac{D}{Dt}$	substantial derivative 2.15
$\frac{D_c}{Dt}$	see Eq. 4.4.7 and 4.4.8
-	vector (over lower case letter) or tensor (over capital letter.)

Section 1

Introduction

This report presents a scalar/vector potential formulation for viscous compressible unsteady flows around complex geometries such as complete aircraft configurations.

Several scalar/vector potential methods are available in the literature for incompressible flows: these methods are reviewed in this Section (with particular emphasis on that proposed by this author). The extension of scalar/vector potential method to compressible flows is presented in Section 4. For the sake of clarity and completeness some classical results on fluid and thermodynamics are presented in Section 2 and 3. It should be emphasized that shock waves and turbulence are not addressed in this report.

1.1. Scalar/Vector Potential Methods for Incompressible Viscous Flows

A review of the state of the art of scalar/vector potential methods for viscous flows is presented here. (This author is not aware of any method for compressible flows, and thus this review is limited to incompressible flows.) In order to discuss the advantages of the approach, the primitive variable approach is also briefly reviewed. Incompressible turbulent viscous flows are governed by the continuity and Navier-Stokes equations. Excellent reviews of the state-of-the-art are given for instance in Refs. 1 and 2 and are summarized here. For the sake of conciseness, boundary layer formulations are not included in this review.

There are two basic approaches to the solution of these equations: solution in the primitive or physical variables (i.e., velocity and pressure) and solution in the scalar-vector potential variables (also called the vorticity-potential method, a three dimensional extension of the two-dimensional vorticity stream-function method). The relative advantages of the two approaches are considered here.

In the primitive-variable formulation there are three basic approaches to the numerical solution of the problem: finite-difference, finite-element and nodal methods (see Refs. 1-3). The major advantage of working with the primitive variables is the simplicity of the equations and the fact that the unknowns have physical meaning. However such formulation has considerable drawbacks. The use of the Navier-Stokes equations requires the

solution in the whole physical space, which may be prohibitive in terms of computer costs. Another difficulty connected with the primitive-variable formulation for incompressible viscous flows is the lack of evolution equation for the pressure: the method of the artificial compressibility introduced to remedy this problem is not fully satisfactory (see Ref. 1). However such a problem does not exist for compressible problems, which are the main objective of this report.

Next consider the scalar/vector potential approach. In two dimensions the advantages of the vorticity/stream-function method over the primitive variable approach are well known. One advantage is that the continuity equation is automatically satisfied. However, a more important advantage, often ignored in literature is that the solution for the equation for vorticity is limited to the computational region of the boundary-layer/wake for attached flow, while the equation for the stream function is a Poisson's equation which can be transformed into an integral equation (also limited to the rotational region). The implications is that the exact solution of the Navier-Stokes equations can be obtained by studying only the rotational region (i.e., boundary layer and wake for attached flows or boundary layer plus separated flow). In other words, the vorticity/stream-function method eliminates all the disadvantages of the formulation in the primitive variables.

To the contrary of a commonly held idea, the vorticity/stream-function approach may be extended to three-dimensional flows. Such a generalization is referred to as the vorticity/potential method or the scalar/vector potential method. The velocity vector \vec{v} is given by the general theorem

$$\vec{v} = \text{grad } \phi + \text{curl } \vec{A} \quad 1.1.1$$

where ϕ is a scalar potential (harmonic for incompressible flows) and \vec{A} is a vector potential. The method is classical: although Lighthill (Refs. 4 and 5) is the standard reference for this approach, according to Lamb (Ref. 6), who gives a theoretical outline of the formulation, this concept was first introduced by Stokes (Ref. 7) and later refined by Helmholtz (Ref. 8). The decomposition is not unique and depends upon the boundary conditions on \vec{A} . This yields the possibility of different 'versions' of the same basic methodology: this issue has been examined very carefully in Ref. 9, which includes an excellent review of the theoretical works on this issue. The theoretical foundation for their work is to be found in the work by Smirnov (Ref. 10). Important in this respect is also the works of Hiraski and Hellums (Refs. 11 and 12), and Ladyzhenskaya (Ref. 13).

The formulation involves solving for the velocity in terms of the vorticity using the law of Biot and Savart. The vorticity is obtained by analyzing the vorticity transport equation, and

determining the vorticity produced at the surface due to the no-slip condition. Since the advent of the computer era, this formulation has been used by several investigators (Refs. 14-24) including the author (Refs. 24-26). Particularly good results have been obtained by Wu and his collaborators (Refs. 18, 21 and 23).

1.2. A Scalar/Vector Potential Method for Incompressible Flows

As mentioned above, all the scalar/vector potential methods are based on the classical decomposition of a vector field into an irrotational component and a rotational (solenoidal) one. The highlights of this version of the method introduced by this author and his collaborators in Refs. 25 and 26 are presented here.

The problem is governed by the averaged Navier-Stokes equations for incompressible flows,

$$\frac{D\bar{\mathbf{v}}}{Dt} = -\frac{1}{\rho} \text{grad } p + \frac{1}{\rho} \mu \nabla^2 \bar{\mathbf{v}} \quad 1.2.1$$

and the continuity equation

$$\text{div } \bar{\mathbf{v}} = 0 \quad 1.2.2$$

The boundary conditions are

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_B \quad 1.2.3$$

where $\bar{\mathbf{v}}_B$ is the velocity of a point on the surface of the body and (for a frame of reference fixed with the undisturbed fluid)

$$\bar{\mathbf{v}} = 0 \quad 1.2.4$$

at infinity.

The method is based on the classical decomposition theorem

$$\bar{\mathbf{v}} = \text{grad } \phi + \text{curl } \bar{\mathbf{A}} \quad 1.2.5$$

where ϕ is a scalar potential and $\bar{\mathbf{A}}$ is a vector potential, related to the vorticity $\bar{\boldsymbol{\zeta}}$ by the relationship

$$\nabla^2 \bar{\mathbf{A}} = -\bar{\boldsymbol{\zeta}} \quad 1.2.6$$

The vorticity is given by the third vortex theorem (obtained by taking the curl of Eq. 1.2.1)

$$\frac{D\bar{\boldsymbol{\zeta}}}{Dt} = (\bar{\boldsymbol{\zeta}} \cdot \text{grad}) \bar{\mathbf{v}} + \frac{\mu}{\rho} \nabla^2 \bar{\boldsymbol{\zeta}} \quad 1.2.7$$

The equation for the potential, obtained by combining Eqs. 1.2.2 and 1.2.5, is

$$\nabla^2 \phi = 0 \quad 1.2.8$$

Equations 1.2.5 to 1.2.7 are fully equivalent to Eqs. 1.2.1 and 1.2.2 and are much easier to solve. First Eq. 1.2.6 yields

$$\bar{A} = \iiint_V \frac{\bar{\omega}}{4\pi r} dV \quad 1.2.9$$

whereas Eq. 1.2.8 can be solved using integral equation methods (also known as panel methods). The numerical formulation is given in Ref. 26.

Finally a brief assessment of the proposed method is given in the following. The advantage of the vorticity/potential method over the primitive-variable approach has already been discussed above (elimination of problem due to lack of pressure-evolution equation, etc.). Here it is important to emphasize again that (1) the method is fully equivalent to the solution of the Navier-Stokes equation and (2) that the solution of Eqs. 1.2.7 and 1.2.9 requires a (finite difference or finite element) grid limited to the nonzero-vorticity region (i.e., boundary layer and wake region for attached flows). However, the main advantage of the formulation presented above is the fact that it may be extended to three-dimensional flows. Such an extension is presented in Section 4 of this report. As mentioned above, this is believed to be the first time that such an extension has been presented.

Section 2

Foundation of Continuum Mechanics

For the sake of clarity and completeness the derivation of the equation of continuum mechanics from fundamental principles is outlined in this section. This derivation is classical and is similar to the one presented in Serrin²⁷. Application to fluids is given in Section 3.

2.1. Basic Definitions

Consider a Cartesian frame of reference in a three-dimensional Euclidean space. Let:

$$\bar{x} = \bar{f}(\xi^a, t) \quad 2.1.1$$

be the (vector) function relating the Cartesian coordinates x^i , (at time t) of a material point identified by convected curvilinear coordinates ξ^a . The coordinates ξ^a could for example coincide with the values of \bar{x} at an arbitrary initial time $t=0$. The function f is assumed to have an inverse

$$\xi^a = F^a(\bar{x}, t) \quad 2.1.2$$

so that there exists a one-to-one correspondence between the point \bar{x} at time t and its curvilinear coordinates. It is assumed that the flow is smooth: in particular surfaces of sharp variations in the velocity, i.e., wakes and shock waves, are not included in the formulation (see Serrin²⁷, pp. 226-228).

An arbitrary quantity g , function of \bar{x} and t , is also a function of ξ^a and t . The following symbols will be used

$$\frac{\partial g}{\partial t} \Big|_{\bar{x} = \text{const}} \quad 2.1.3$$

$$\frac{Dg}{Dt} \Big|_{\xi^a = \text{const}} \quad 2.1.4$$

Dg/Dt is called the material (or substantial) derivative of g . Note, that by using the chain rule:

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \sum_i \frac{\partial g}{\partial x^i} \frac{\partial x^i}{\partial t} \Big|_{\xi^a = \text{const}} \quad 2.1.5$$

or

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \bar{v} \cdot \text{grad } g \quad 2.1.6$$

where \bar{v} is the velocity of the material point

$$\bar{v}^i = \frac{\partial x^i}{\partial t} \Big|_{\xi^a = \text{const}} = \frac{Dx^i}{Dt} \quad 2.1.7$$

2.2. Fundamental Conservation Principles

The motion of the continuum is assumed to be governed by the following fundamental principles:

Conservation of Mass

$$\frac{d}{dt} \iiint_{V_M} \rho dV = 0 \quad 2.2.1$$

Conservation of Momentum

$$\frac{d}{dt} \iiint_{V_M} \rho \bar{v} dV = \iiint_{V_M} \rho \bar{f} dV + \oint_{\mathcal{G}} \bar{t} d\mathcal{G} \quad 2.2.2$$

Conservation of Angular Momentum

$$\frac{d}{dt} \iiint_{V_M} \rho \bar{x} \times \bar{v} dV = \iiint_{V_M} \rho \bar{x} \times \bar{f} dV + \oint_{\mathcal{G}} \bar{x} \times \bar{t} d\mathcal{G} \quad 2.2.3$$

Conservation of Energy (or the first law of thermodynamics, Ref. 2, p. 177)

$$\frac{d}{dt} \iiint_{V_M} \rho \left(e + \frac{1}{2} \bar{v} \cdot \bar{v} \right) dV = \iiint_{V_M} \rho \bar{f} \cdot \bar{v} dV + \oint_{\mathcal{G}} \bar{t} \cdot \bar{v} d\mathcal{G} - \oint_{\mathcal{G}} Q d\mathcal{G} \quad 2.2.4$$

In Equations 2.2.1, 2.2.2, 2.2.3 and 2.2.4 the volume V_M is an arbitrary material volume (i.e., by definition a volume which moves with the continuum particles), \bar{v} is the velocity of the fluid particles defined by Eq. 2.1.7, ρ is the density (mass per unit volume), \bar{f} is the force per unit mass acting on the fluid at a point of the volume V_M , \bar{t} is the stress vector (or force per unit surface area) acting on the continuum at a point of the surface \mathcal{G} surrounding V_M and h is the heat flux supplied by the volume V_M .

It may be worth noting that, in postulating the above principles as the governing equations of the motion of the continuum, it is implicitly assumed that we are dealing with simple nonreacting species; multiphase flows, and/or chemical reactions are not included in the formulation. Also electromagnetic phenomena are not considered in this report.

A considerable amount of results can be obtained as a consequence of the above set of principles. The rest of this section is devoted to the derivation of such results.

2.3. Jacobian of Transformation

Note that, for any arbitrary function g ,

$$\iiint_{V_M} g dV = \iiint_{V_0} g J d\xi^1 d\xi^2 d\xi^3 \quad 2.3.1$$

where V_0 is the image of V_M in the ξ^a space, J is the

Jacobian of the transformation

$$J = \det \left(\frac{\partial x^i}{\partial \xi^\alpha} \right) > 0 \quad 2.3.2$$

For future reference, note that:

$$\frac{DJ}{Dt} = J \operatorname{div} \bar{v} \quad 2.3.3$$

In order to prove this, note that if X_i^α is the cofactor of $\partial x^i / \partial \xi^\alpha$ so that

$$\sum_\alpha \frac{\partial x^i}{\partial \xi^\alpha} X_j^\alpha = J \delta_j^i \quad 2.3.4$$

then,

$$\begin{aligned} \frac{DJ}{Dt} &= \sum_{i,\alpha} \frac{D}{Dt} \left(\frac{\partial x^i}{\partial \xi^\alpha} \right) X_i^\alpha = \sum_{i,\alpha} \frac{\partial v^i}{\partial \xi^\alpha} X_i^\alpha \\ &= \sum_{i,j,\alpha} \frac{\partial v^i}{\partial x^j} \frac{\partial x^j}{\partial \xi^\alpha} X_i^\alpha = \sum_i \frac{\partial v^i}{\partial x^i} J = J \operatorname{div} \bar{v} \end{aligned} \quad 2.3.5$$

2.4. Time Derivative of Integrals

Note that if V_M is a volume moving with the fluid, then the boundaries of its image V_0 in the ξ^α space are time independent. Therefore, using Eqs. 2.3.1 and 2.3.3 one obtains:

$$\frac{d}{dt} \iiint_{V_M} g dV = \iiint_{V_M} \left(\frac{Dg}{Dt} + g \operatorname{div} \bar{v} \right) dV \quad 2.4.1$$

For,

$$\begin{aligned} \frac{d}{dt} \iiint_{V_M} g dV &= \iiint_{V_0} \frac{D}{Dt} (gJ) d\xi^1 d\xi^2 d\xi^3 \\ &= \iiint_{V_0} \left(\frac{Dg}{Dt} J + g \frac{DJ}{Dt} \right) d\xi^1 d\xi^2 d\xi^3 \\ &= \iiint_{V_0} \left(\frac{Dg}{Dt} + g \operatorname{div} \bar{v} \right) J d\xi^1 d\xi^2 d\xi^3 \end{aligned} \quad 2.4.2$$

Also, note that

$$\frac{Dg}{Dt} + g \operatorname{div} \bar{v} \equiv \frac{\partial g}{\partial t} + \operatorname{div}(g \bar{v}) \quad 2.4.3$$

hence, applying the divergence theorem

$$\iiint_{V_M} \operatorname{div} \bar{w} dV = \oint_{\bar{\sigma}} \bar{w} \cdot \bar{n} d\sigma \quad 2.4.4$$

(where \bar{n} is the outer normal to the surface σ) one obtains

$$\frac{d}{dt} \iiint_{V_M} g dV = \iiint_{V_M} \frac{\partial g}{\partial t} dV + \oint_{\bar{\sigma}} g \bar{v} \cdot \bar{n} d\sigma \quad 2.4.5$$

which states that the rate of change of the volume integral is due to the rate of change of the integrand over a fixed volume plus the flux over the boundary surface.

2.5. Continuity Equation

Consider the principle of conservation of mass, Eq. 2.2.1. Using Eq. 2.4.2, Eq. 2.2.1 yields (noting that V_M is an arbitrary volume)

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{v} = 0 \quad 2.5.1$$

or, according to Eq. 2.4.3,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \bar{v}) = 0 \quad 2.5.2$$

Also combining Eq. 2.3.3 and 2.5.1

$$\frac{D}{Dt}(\rho J) = 0 \quad 2.5.3$$

which can be obtained directly from Eq. 2.2.1, using Eq. 2.3.1 and 2.4.2 and noting that the volume V_M is arbitrary. Equation 2.5.3 yields:

$$\rho J = \text{constant} = \rho_0 J_0 \quad (\text{following particle}) \quad 2.5.4$$

Equations 2.5.1, 2.5.2, 2.5.3, and 2.5.4 are four different forms of the continuity equation. Note that, using Eq. 2.5.3 one obtains, for any function g ,

$$\frac{d}{dt} \iiint_{V_M} \rho g dV = \iiint_{V_M} \rho \frac{Dg}{Dt} dV \quad 2.5.5$$

For,

$$\frac{d}{dt} \iiint_{V_M} \rho g dV = \iiint_{V_0} \frac{Dg}{Dt} \rho J d\xi^1 d\xi^2 d\xi^3 \quad 2.5.6$$

2.6. Stress Tensor

Consider the principle of conservation of momentum, Eq. 2.2.2. applied to an infinitesimal volume V_M . Let ℓ be a typical length of the volume V_M so that

$$V_M = O(\ell^3) \quad 2.6.1$$

whereas

$$\sigma = O(\ell^2) \quad 2.6.2$$

Letting ϵ go to zero, one obtains, in the limit

$$\oint_{\epsilon} \bar{t} d\epsilon = 0 \quad 2.6.3$$

Assume that V_{ϵ} coincides with an infinitesimal Cauchy's tetrahedron, i.e., a tetrahedron with the origin at an arbitrary point \bar{x} and three faces parallel to the coordinate planes (i.e., having outward unit normals $-\bar{i}_1, -\bar{i}_2, -\bar{i}_3$) and the fourth face in the first octant with normal \bar{n} (see Figure 2.1). In this case Eq. 2.6.3 yields

$$\bar{t}(\bar{n}) d\epsilon + \bar{t}(-\bar{i}_1) d\epsilon_1 + \bar{t}(-\bar{i}_2) d\epsilon_2 + \bar{t}(-\bar{i}_3) d\epsilon_3 = 0 \quad 2.6.4$$

or setting $\bar{t}(-\bar{i}_k) = -\bar{t}(\bar{i}_k) = -\bar{t}_k$ and noting that $d\epsilon_k = n_k d\epsilon$, (where n_k are the components of the unit normal \bar{n}) one obtains

$$\bar{t} = \bar{t}_1 n_1 + \bar{t}_2 n_2 + \bar{t}_3 n_3 \quad 2.6.5$$

i.e.,

$$t_k = \sum_j n_j T_{jk} \quad 2.6.6$$

where T_{jk} is the k th component of $\bar{t}_j = \bar{t}(\bar{i}_j)$.

The above result can be stated as follows: the forces acting on three coordinate surfaces through a given point defines a tensor (stress tensor), with components T_{jk} : the force acting on any surface normal to a given direction \bar{n} is dependent upon these quantities through Eq. 2.6.6, which, in tensor notations, may be rewritten as

$$\bar{t} = \bar{n} \cdot \bar{T} \quad 2.6.7$$

2.7. Cauchy's Equation of Motion

Note that, according to Eq. 2.6.6, and using the divergence theorem

$$\begin{aligned} \oint_{\epsilon} \bar{t} d\epsilon &= \sum_{j,k} \oint_{\epsilon} n_j T_{jk} \bar{i}_k d\epsilon \\ &= \sum_{j,k} \iiint_V \frac{\partial T_{jk}}{\partial x_j} \bar{i}_k dV \\ &= \iiint_V \text{div } \bar{T} dV \end{aligned} \quad 2.7.1$$

where $\text{div } \bar{T}$ is a vector defined as

$$\text{div } \bar{T} = \sum_{j,k} \frac{\partial T_{jk}}{\partial x_j} \bar{i}_k \quad 2.7.2$$

Combining the conservation of momentum, Eq. 2.2.2 with Eq. 2.7.1 and using Eq. 2.5.5 one obtains

$$\frac{d}{dt} \iiint_{V_M} \rho \bar{v} dV = \iiint_{V_M} \rho \frac{\partial \bar{v}}{\partial t} dV = \iiint_{V_M} (\rho \bar{f} + \text{div } \bar{T}) dV \quad 2.7.3$$

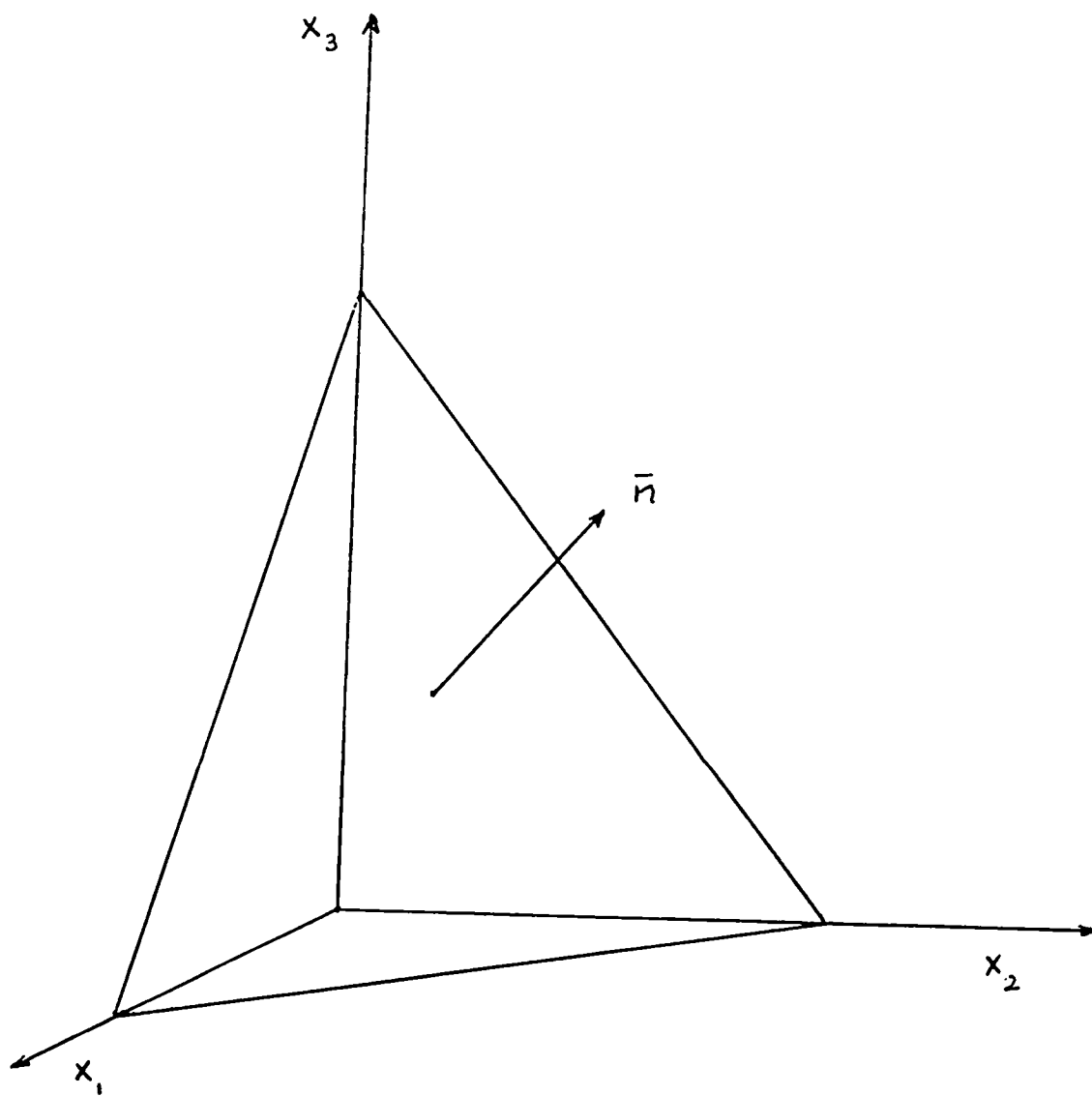


Figure 2.1 Cauchy tetrahedron

or, noting that V_M is arbitrary,

$$\rho \frac{D\bar{v}}{Dt} = \rho \bar{f} + \text{div } \bar{T} \quad 2.7.4$$

which is called Cauchy's equation of motion (Serrin²⁷, p. 135) or dynamic equilibrium equation.

2.8. Symmetry of Stress Tensor

Note, that according to Eq 2.6.6,

$$\begin{aligned} \oint_G \bar{x} x \bar{t} \, d\sigma &= \sum_{j,k} \oint_G \bar{x} x (T_{jk} n_j \bar{i}_k) \, d\sigma = \sum_{j,k} \iiint_{V_M} \frac{\partial}{\partial x_j} (\bar{x} x T_{jk} \bar{i}_k) \, dV \\ &= \sum_{j,k} \iiint_{V_M} \bar{x} x \frac{\partial}{\partial x_j} T_{jk} \bar{i}_k \, dV + \sum_{j,k} \iiint_{V_M} \bar{i}_j x \bar{i}_k T_{jk} \, dV \\ &= \iiint_{V_M} \bar{x} x \text{div } \bar{T} \, dV + \sum_{j,k} \iiint_{V_M} T_{jk} \bar{i}_j x \bar{i}_k \, dV \end{aligned} \quad 2.8.1$$

Combining the conservation of angular momentum Eq. 2.2.3, with Eqs 2.5.5 and 2.8.1 one obtains (noting that $\bar{v} x D\bar{r}/Dt = \bar{v} x \bar{v} = 0$)

$$\begin{aligned} \frac{d}{dt} \iiint_{V_M} \rho \bar{x} x \bar{v} \, dV &= \iiint_{V_M} \rho \frac{D}{Dt} (\bar{x} x \bar{v}) \, dV = \iiint_{V_M} \rho \bar{x} x \frac{D\bar{v}}{Dt} \, dV \\ &= \iiint_{V_M} \rho \bar{x} x \bar{f} \, dV + \iiint_{V_M} \bar{x} x \text{div } \bar{T} \, dV + \sum_{j,k} \iiint_{V_M} T_{jk} \bar{i}_j x \bar{i}_k \, dV \end{aligned} \quad 2.8.2$$

or, using the equilibrium equation, Eq. 2.7.4, and noting that V_M is arbitrary

$$\sum_{j,k} T_{jk} \bar{i}_j x \bar{i}_k = 0 \quad 2.8.3$$

or

$$T_{jk} = T_{kj} \quad 2.8.4$$

which shows that the stress tensor is symmetric.

2.9. Energy Transfer Equation and Virtual Work Principle

Consider the equilibrium equation, Eq. 2.7.4. Taking the dot product of both sides of the equation with \bar{v} and integrating over an arbitrary volume V_M one obtains:

$$\iiint_{V_M} \rho \frac{D\bar{v}}{Dt} \cdot \bar{v} \, dV = \iiint_{V_M} \rho \bar{f} \cdot \bar{v} \, dV + \iiint_{V_M} \text{div } \bar{T} \cdot \bar{v} \, dV \quad 2.9.1$$

Note that, using Eqs. 2.6.6, 2.7.2, and 2.8.4,

$$\begin{aligned}
\iiint_{V_M} \operatorname{div} \bar{T} \cdot \bar{v} \, dV &= \sum_{j,k} \iiint_{V_M} \frac{\partial \bar{T}_{jk}}{\partial x_j} \bar{i}_k \cdot \bar{v} \, dV \\
&= \sum_{j,k} \iiint_{V_M} \frac{\partial T_{jk}}{\partial x_j} v_k \, dV = \\
&= \sum_{j,k} \iiint_{V_M} \frac{\partial}{\partial x_j} (T_{jk} v_k) \, dV - \sum_{j,k} \iiint_{V_M} T_{jk} \frac{\partial v_k}{\partial x_j} \, dV \\
&= \sum_{j,k} \oint_{\sigma} T_{jk} v_k n_j \, d\sigma - \sum_{j,k} \frac{1}{2} \iiint_{V_M} T_{jk} \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right) \, dV \\
&= \oint_{\sigma} \bar{t} \cdot \bar{v} \, d\sigma - \iiint_{V_M} \bar{T} : \bar{D} \, dV \quad 2.9.2
\end{aligned}$$

where \bar{D} is a symmetric tensor with component

$$D_{jk} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right) \quad 2.9.3$$

and is called the deformation tensor (Serrin, p. 139) or strain-rate tensor (see also section 2.B).

The last integral in Eq. 2.9.2 is called dissipation term (Ref.27, p. 138), and is the work per unit time done by the internal stresses.

Combining Eqs. 2.9.1 and 2.9.2 yields, if V_M is a material volume:

$$\begin{aligned}
\frac{d}{dt} \iiint_{V_M} \frac{1}{2} \rho \bar{v} \cdot \bar{v} \, dV &= \iiint_{V_M} \rho \bar{f} \cdot \bar{v} \, dV + \oint_{\sigma} \bar{t} \cdot \bar{v} \, d\sigma \\
&\quad - \iiint_{V_M} \bar{T} : \bar{D} \, dV \quad 2.9.4
\end{aligned}$$

which states that the time derivative of the kinetic energy of a material volume is equal to the work per unit time of the volume and surface forces diminished by dissipation term.

The above result is called energy transfer equation or mechanical energy equation and relates 'mechanical energy terms' to the dissipation term.

It may be worth noting that if Eq. 2.7.4 multiplied (internal product) by $\delta \bar{u}$ instead of \bar{v} one obtains the virtual-work equation (which here is not assumed as an independent principle)

$$\iiint_{V_M} \rho \frac{D\bar{v}}{Dt} \cdot \delta \bar{u} \, dV = \iiint_{V_M} \rho \bar{f} \cdot \delta \bar{u} \, dV + \oint_{\sigma} \bar{t} \cdot \delta \bar{u} \, d\sigma - \iiint_{V_M} \bar{T} : \delta \bar{U} \, dV \quad 2.9.5$$

where

$$\delta U_{jk} = \frac{\partial}{\partial x_j} \delta u_k \quad 2.9.6$$

2.10. Thermodynamic Energy Equation

Comparing the principle of conservation of energy, Eq. 2.2.4 with the energy transfer equation, Eq. 2.9.4, one obtains:

$$\frac{d}{dt} \iiint_{V_M} \rho e dV = \iiint_{V_M} \rho \frac{De}{Dt} dV = \iiint_{V_M} \bar{T} : \bar{D} dV - \oint_{\bar{\sigma}} Q d\sigma \quad 2.10.1$$

which relates thermodynamic energy terms to the dissipation term. For this reason Eq. 2.10.1 is here called the integral thermodynamic energy equation.

It may be worth noting that Eq. 2.10.1 is often referred to as the first principle of thermodynamics. Here we assume Eq. 2.2.4 (conservation of energy, as a fundamental principle. Eq. 2.10.1 is a consequence of Eq. 2.2.4, not a fundamental principle.

2.11. Heat Flux Vector

Note that in Eq. 2.10.1 if ℓ is a typical size of the volume V and ℓ goes to zero then the surface integral is of order ℓ^2 whereas the other terms are of order ℓ^3 . Hence if V_M is an infinitesimal Cauchy's tetrahedron (see Figure 2.1) one obtains (see Section 2.6) that:

$$Q(\bar{n}) = Q(\bar{i}_1)n_1 + Q(\bar{i}_2)n_2 + Q(\bar{i}_3)n_3 \quad 2.11.1$$

or

$$Q = \bar{q} \cdot \bar{n} \quad 2.11.2$$

where \bar{q} is a vector with components:

$$q_k = Q(\bar{i}_k) \quad 2.11.3$$

equal to the heat flux per unit area through the surface normal to \bar{i}_k .

2.12. Thermodynamic Energy Equation in Differential Form

Combining Eq. 2.11.3 with the thermodynamic energy equation Eq. 2.10.1, applying divergence theorem and noting that the volume is arbitrary yields:

$$\rho \frac{De}{Dt} = \bar{T} : \bar{D} - \text{div } \bar{q} \quad 2.12.1$$

which will be called here the differential form of the thermodynamic energy equation. 'According to Truesdell this equation should be attributed to C. Newman' (Serrin², p.177)

2.A. A Convenient Expression for the Acceleration

Consider

$$\frac{Dv_k}{Dt} = \frac{\partial v_k}{\partial t} + \sum_j v_j \frac{\partial v_k}{\partial x_j} \quad 2.A.1$$

Note that

$$\sum_j v_j \frac{\partial v_k}{\partial x_j} = \sum_j v_j \frac{\partial v_j}{\partial x_k} + \sum_j v_j \left(\frac{\partial v_k}{\partial x_j} - \frac{\partial v_j}{\partial x_k} \right) \quad 2.A.2$$

and set

$$\bar{\zeta} = \text{curl } \bar{v} \quad 2.A.3$$

so that

$$\frac{D\bar{v}}{Dt} = \frac{\partial \bar{v}}{\partial t} + \frac{1}{2} \text{grad } v^2 - \bar{v} \times \bar{\zeta} \quad 2.A.4$$

and Cauchy's law of motion may be written as

$$\frac{\partial \bar{v}}{\partial t} + \frac{1}{2} \text{grad } v^2 - \bar{v} \times \bar{\zeta} = \bar{f} + \frac{1}{\rho} \text{div } \bar{T} \quad 2.A.5$$

2.B. Kinematics of Deformation

The relationship between the vorticity and the angular velocity of a fluid element surrounding a point \bar{x}_0 is introduced in this section.

Consider the Taylor series of the velocity field about any arbitrary fixed point \bar{x}_0 in the flow field

$$v_j(\bar{x}) = v_j(\bar{x}_0) + \sum_k \frac{\partial v_j}{\partial x_k} r_k + O(r^2) \quad 2.B.1$$

where

$$\bar{r} = \bar{x} - \bar{x}_0 \quad 2.B.2$$

Equation 2.B.1. may be rewritten as

$$\bar{v} = \bar{v}_0 + \bar{r} \cdot \bar{\Omega} + \bar{r} \cdot \bar{D} \quad 2.B.3$$

where the tensor $\text{grad } \bar{v}$ has been decomposed into its symmetric and antisymmetric parts

$$\text{grad } \bar{v} = \bar{D} + \bar{\Omega} \quad 2.B.4$$

where \bar{D} is the (symmetric) deformation tensor with components

$$D_{jk} = \frac{1}{2} \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right) \quad 2.B.5$$

(see Eq. 2.9.3), whereas $\bar{\Omega}$ is the (antisymmetric) rotation tensor with components

$$\Omega_{kj} = \frac{1}{2} \left(\frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_j} \right) \quad 2.B.6$$

Note that

$$\bar{r} \cdot \bar{\Omega} = \sum_{kj} \Omega_{kj} \bar{r}_k \bar{i}_j = \bar{\omega} \times \bar{r} \quad 2.B.7$$

where

$$\bar{\omega} = (\Omega_{23}, \Omega_{31}, \Omega_{12}) = \frac{1}{2} \text{curl } \bar{v} = \frac{1}{2} \bar{\zeta} \quad 2.B.8$$

Also

$$\begin{aligned} \bar{r} \cdot \bar{D} &= \sum_{kj} D_{kj} \bar{r}_k \bar{i}_j \\ &= \frac{1}{2} \text{grad } D \end{aligned} \quad 2.B.9$$

where

$$D = \sum_{jk} \bar{r}_j D_{jk} \bar{r}_k = \bar{r} \cdot \bar{D} \cdot \bar{r} \quad 2.B.10$$

is called strain rate quadric. Hence Eq. 2.B.3 may be rewritten as

$$\bar{v} = \bar{v}_0 + \bar{\omega}_0 \times \bar{r} + \bar{v}_D \quad 2.B.11$$

with

$$\bar{v}_D = \frac{1}{2} \text{grad } D \quad 2.B.12$$

which indicates that the motion of a fluid element around a point \bar{x}_0 can be decomposed in translation (with velocity $\bar{v}_0 = \bar{v}(\bar{x}_0)$), rotation (with angular velocity $\bar{\omega}_0$ equal to half the vorticity, $\text{curl } \bar{v}$, at \bar{x}_0) and deformation (with deformation velocity equal to $\text{grad } D/2$).

Section 3

Entropy and the Second Law of Thermodynamics

The fundamental equations of continuum mechanics were derived in Section 2, from the fundamental principles of conservation of mass, momentum, angular momentum and energy in the form of Eqs. 2.2.1, 2.2.2, 2.2.3, and 2.2.4. In this section the formulation is carried further by assuming that the continuum be a fluid: the equation of state for a fluid is postulated along with the second law of thermodynamics. The formulation is again classical and the presentation given here is quite similar to the one of Serrin¹⁷ (pp. 172-178 and 230-241).

3.1. Starting Equations

Starting from the basic principles of conservation the following equations were derived in Section 2:

Continuity equation (Eq. 2.5.1)

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{\mathbf{v}} = 0 \quad 3.1.1$$

Cauchy's equation of motion (Eq. 2.7.4.)

$$\rho \frac{D\bar{\mathbf{v}}}{Dt} = \rho \bar{\mathbf{f}} + \operatorname{div} \bar{\mathbf{T}} \quad 3.1.2$$

Symmetry of stress tensor (Eq. 2.8.4)

$$\mathbf{T}_{jk} = \mathbf{T}_{kj} \quad 3.1.3$$

Thermodynamic energy equation (Eq. 2.12.1)

$$\rho \frac{De}{Dt} = \bar{\mathbf{T}} : \bar{\mathbf{D}} - \operatorname{div} \bar{\mathbf{q}} \quad 3.1.4$$

where the deformation tensor, $\bar{\mathbf{D}}$, is defined by (Eq. 2.9.3)

$$D_{jk} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right) \quad 3.1.5$$

In Eq. 3.1.2, the force per unit mass $\bar{\mathbf{f}}$ is prescribed. It is apparent that if $\bar{\mathbf{T}}$ and $\bar{\mathbf{q}}$ were known, then Eqs. 3.1.1, 3.1.2 and 3.1.4, with appropriate initial conditions and boundary conditions, could be used to obtain ρ , $\bar{\mathbf{v}}$ and e . In other words, the above equations cannot be used to solve the problem unless suitable constitutive relations (relating $\bar{\mathbf{T}}$ and $\bar{\mathbf{q}}$ to other quantities) are available. The objective of this section is to show that the second law of thermodynamics (introduced in Section 3.4) puts a constraint (Eq. 3.4.4) on the general nature of the constitutive equations. The actual constitutive equations are introduced in Section 3.B.

3.2. Equation of State

Our continuum is assumed to be a fluid, i.e., a single-phase system which is described by two state variables, for instance internal energy e and density ρ introduced in Section 2. All other variables are assumed to be functions of the first two. Any two variables however may be chosen as the main variables: 'A particularly elegant formulation of this relation is that of Gibbs' (Serrin²⁷, p.172), which is followed here: the two main state variables are chosen to be the entropy (which is assumed as a primitive concept, like the energy), and the specific volume, $\tau=1/\rho$. The fundamental state equation is some definite relationship giving the internal energy as a function of entropy and specific volume, of the type

$$e = e(S, \tau) \quad 3.2.1$$

In addition, the (thermodynamic) pressure and the temperature are defined by

$$p = - \frac{\partial e}{\partial \tau} \quad 3.2.2$$

$$T = \frac{\partial e}{\partial S} \quad 3.2.3$$

(it is assumed, of course, that p and T are greater than zero). Since e and $\rho=1/\tau$ have already been introduced, Eq. 3.2.1 could be thought of as the (implicit) definition of entropy. Note also that the pressure is introduced as a thermodynamic quantity (see also Eq. 3.3.4) rather than a mechanical one (i.e., force per unit surface).

The differential of the internal energy is

$$de = TdS - p d\tau \quad 3.2.4$$

and, accordingly, the material time derivative is given by

$$\frac{De}{Dt} = T \frac{DS}{Dt} - p \frac{D\tau}{Dt} \quad 3.2.5$$

3.3. Entropy Evolution Equation

Combining the thermodynamic energy equation, Eq. 3.1.4, with Eq. 3.2.5 one obtains

$$\rho T \frac{DS}{Dt} - \rho p \frac{D\tau}{Dt} = \bar{T} : \bar{D} - \text{div } \bar{q} \quad 3.3.1$$

Note that $\tau=1/\rho$ and therefore, according to Eq. 3.1.1

$$\rho \frac{D\tau}{Dt} = - \frac{1}{\rho} \frac{D\rho}{Dt} = \text{div } \bar{v} \quad 3.3.2$$

Combining Eq. 3.3.1 and Eq. 3.3.2 yields

$$\rho T \frac{DS}{Dt} = p \operatorname{div} \bar{v} + \bar{T} : \bar{D} - \operatorname{div} \bar{q} \quad 3.3.3$$

Equation 3.3.2 may be rewritten in simpler form if we introduce the tensor, \bar{V} , defined by

$$\bar{V} = \bar{T} + p \bar{I} \quad 3.3.4$$

where p is the thermodynamic pressure (Eq. 3.2.2) and \bar{I} is the unit tensor. Using Eq. 3.3.4, Eq. 3.3.3 may be written as

$$\rho T \frac{DS}{Dt} = \bar{\Phi} - \operatorname{div} \bar{q} \quad 3.3.5$$

where

$$\bar{\Phi} = \bar{V} : \bar{D} = \bar{T} : \bar{D} + p \operatorname{div} \bar{v} \quad 3.3.6$$

Equation 3.3.5 will be referred to as the entropy-evolution equation. It should be noted that Eq. 3.3.5 was obtained from the total energy equation (Eq. 3.1.4), by replacing the internal energy, e , with its expression obtained from the fundamental state equation. Therefore Eq. 3.3.5 is fully equivalent to the principle of conservation of energy.

3.4. Second Law of Thermodynamics

The main postulate introduced in this section is the second law of thermodynamics:

$$\frac{d}{dt} \iiint_{V_M} \rho S dV > - \oint_{\sigma} \frac{1}{T} \bar{q} \cdot \bar{n} d\sigma \quad 3.4.1$$

where V_M is a fluid-volume (i.e., by definition, a volume moving with the fluid particles, see Section 2.2).

Equation 3.4.1, may be rewritten in a much more interesting form, by noting that, using the entropy-evolution equation, Eq. 3.3.5 (see also Eq. 2.5.5), one obtains:

$$\begin{aligned} \frac{d}{dt} \iiint_{V_M} \rho S dV &= \iiint_{V_M} \rho \frac{DS}{Dt} dV \\ &= \iiint_{V_M} \left(\frac{\bar{\Phi}}{T} - \frac{1}{T} \operatorname{div} \bar{q} \right) dV \\ &= \iiint_{V_M} \left(\frac{\bar{\Phi}}{T} - \frac{1}{T^2} \bar{q} \cdot \operatorname{grad} T \right) dV - \oint_{\sigma} \frac{1}{T} \bar{q} \cdot \bar{n} d\sigma \end{aligned} \quad 3.4.2$$

Hence the second law of thermodynamics, Eq. 3.4.1, may be rewritten as:

$$\begin{aligned} \frac{d}{dt} \iiint_{V_M} \rho S dV + \oint_{\sigma} \frac{1}{T} \bar{q} \cdot \bar{n} d\sigma = \\ = \iiint_{V_M} \left(\frac{\bar{\Phi}}{T} - \frac{1}{T^2} \bar{q} \cdot \text{grad } T \right) dV > 0 \end{aligned} \quad 3.4.3$$

or, since V_M is arbitrary,

$$\bar{\Phi} - \frac{1}{T} \bar{q} \cdot \text{grad } T > 0 \quad 3.4.4$$

(this equation will be used to discuss the constitutive relations, Section 4.A).

It is instructive to rewrite the total energy equation, Eq. 3.1.4, as (see Eq. 3.3.6)

$$\rho \frac{De}{Dt} + p \text{div } \bar{v} = - \text{div } \bar{q} + \bar{\Phi} \quad 3.4.5$$

which shows on the right hand-side of the dissipative terms, whereas the terms on the left hand side are nondissipative. Both terms are equal to $\rho T DS/Dt$ (Eq. 3.3.5), which indicates that the entropy is the 'link' between dissipative and nondissipative terms.

3.5. Thermodynamic Pressure and Mechanical Pressure

It is worth noting that Eq. 3.3.4 may be rewritten as

$$\bar{T} = - p \bar{I} + \bar{V} \quad 3.5.1$$

where p is still the thermodynamic quantity originally defined by Eq. 3.2.2. However it is apparent that in Eq. 3.5.1, p assumes the role of mechanical pressure (force per unit area). This is quite clearly indicated by the fact that, in Eq. 3.4.4, $\bar{\Phi}$ (see Eq. 3.3.6) is affected by \bar{V} , not by \bar{T} . Therefore the second law of thermodynamics itself suggests that \bar{T} be given in the form of Eq. 3.5.1 (with \bar{V} responsible for the dissipative effects), i.e., that the nondissipative part of \bar{T} be given by $-p\bar{I}$, where p is the thermodynamic pressure (in particular for perfect fluids, $\bar{T} = -p\bar{I}$). This point, clarified further in Section 4B, is somewhat obscure in the literature where Eq. 3.5.1 is introduced as an independent assumption.

3.6. Enthalpy

Note that, using Eq. 3.5.1, Cauchy's equation of motion may be rewritten as

$$\frac{D\bar{v}}{Dt} = \bar{f} - \frac{1}{\rho} \text{grad } p + \frac{1}{\rho} \text{div } \bar{V} \quad 3.6.1$$

In discussing Bernoullian theorems, it is convenient to express $(1/\rho) \text{ grad } p$ in terms of a conservative (i.e., exact differential) and nonconservative part: note that according to Eq. 3.2.4

$$\begin{aligned}\frac{1}{\rho} dp &= \tau dp = d(\tau p) - p d\tau \\ &= d(e + \tau p) - T dS\end{aligned}\quad 3.6.2$$

and hence

$$\frac{1}{\rho} \text{grad } p = \text{grad } h - T \text{grad } S \quad 3.6.3$$

where the enthalpy h is defined as

$$h = e + \frac{p}{\rho} \quad 3.6.4$$

Accordingly Eq. 3.6.1 may be rewritten as

$$\frac{D\bar{v}}{Dt} = \bar{f} - \text{grad } h + T \text{grad } S + \frac{1}{\rho} \text{div } \bar{V} \quad 3.6.5$$

Note that for barotropic flows

$$\frac{1}{\rho} \text{grad } p = \text{grad } \int_0^p \frac{dp}{\rho} \quad 3.6.6$$

Here Eq. 3.6.3 has the same 'role' that Eq. 3.6.6 has for barotropic-flow formulations (for isentropic flows Eq. 3.6.3 and Eq. 3.6.6 coincide since, in this case $dh=dp/\rho$).

3.A Dependence of Internal Energy on Specific Volume

A classical result on $\partial u/\partial \tau$ for $T = \text{constant}$ is derived here for use in Section 4.A. Equation 3.2.4 may be rewritten as

$$dS = \frac{1}{T} de + \frac{p}{T} d\tau \quad 3.A.1$$

or, using T and τ as fundamental variables

$$dS = \frac{1}{T} \frac{\partial e}{\partial T} dT + \frac{1}{T} \frac{\partial e}{\partial \tau} d\tau + \frac{p}{T} d\tau \quad 3.A.2$$

Hence,

$$\frac{\partial S}{\partial T} = \frac{1}{T} \frac{\partial e}{\partial T} \quad 3.A.3$$

$$\frac{\partial S}{\partial \tau} = \frac{1}{T} \left(\frac{\partial e}{\partial \tau} + p \right) \quad 3.A.4$$

where it is understood that the partial differentiation with respect to T is performed with $\tau = \text{const}$ (and viceversa). Therefore

$$\begin{aligned}
\frac{\partial^2 S}{\partial \tau \partial T} &= \frac{\partial}{\partial \tau} \left(\frac{1}{T} \frac{\partial e}{\partial T} \right) = \frac{1}{T} \frac{\partial^2 e}{\partial \tau \partial T} \\
&= \frac{\partial^2 S}{\partial T \partial \tau} = \frac{\partial}{\partial T} \left(\frac{1}{T} \frac{\partial e}{\partial \tau} + \frac{p}{T} \right) \\
&= -\frac{1}{T^2} \left(\frac{\partial e}{\partial \tau} + p \right) + \frac{1}{T} \left(\frac{\partial^2 e}{\partial T \partial \tau} + \frac{\partial p}{\partial T} \right)
\end{aligned} \tag{3.A.5}$$

or

$$\left. \frac{\partial e}{\partial \tau} \right|_T = T \left. \frac{\partial p}{\partial T} \right|_{\tau} - p \tag{3.A.6}$$

Equation 3.A.6 indicates that

$$\left. \frac{\partial e}{\partial \tau} \right|_T = 0 \tag{3.A.7}$$

whenever, for $\tau = \text{constant}$, p is a linear function of T .

Section 4

Vector and Scalar Potentials

The fundamental equations governing the motion of a fluid were derived in Sections 2 and 3. In this section the fundamental equations will be rewritten to obtain a formulation in terms of vector and scalar potentials.

4.1. Starting Equations

The following equations are used in this section:

Continuity equation (Eq. 3.1.1)

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{v} = 0 \quad 4.1.1$$

Cauchy's equilibrium equation (Eq. 3.6.5)

$$\frac{D\bar{v}}{Dt} = \bar{f} - \operatorname{grad} h + T \operatorname{grad} S + \frac{1}{\rho} \operatorname{div} \bar{V} \quad 4.1.2$$

Entropy evolution equation (Eq. 3.3.1)

$$\rho T \frac{DS}{Dt} = \bar{\Phi} - \operatorname{div} \bar{q} \quad 4.1.3$$

Also the expression for the acceleration given by Eq. 2.A.4

$$\frac{D\bar{v}}{Dt} = \frac{\partial \bar{v}}{\partial t} + \frac{1}{2} \operatorname{grad} v^2 - \bar{v} \times \bar{\zeta} \quad 4.1.4$$

will be used.

It is understood that the equations of state, Eqs. 3.2.1, 3.2.2 and 3.2.3 are available. It is also assumed that some constitutive relations (which give \bar{T} and \bar{q} in terms of other quantities) are available. Such relationships are discussed in Section 4A. The reason the introduction of the constitutive relations is postponed is to emphasize the generality of the vector/scalar-potential formulation, i.e., that the derivation of such a formulation is independent of the specific expressions used for \bar{T} and \bar{q} .

4.2. Decomposition Theorem

A fundamental theorem of vector field theory states that any vector field, in particular, in our case, the velocity field, can be decomposed as (see Serrin, p. 164-165)

$$\bar{v} = \operatorname{grad} \phi + \operatorname{curl} \bar{A} \quad 4.2.1$$

where ϕ is called the scalar potential, whereas \bar{A} satisfies the relation

$$\text{div } \bar{A} = 0 \quad 4.2.2$$

and is called the vector potential.

Note that the decomposition is not unique since any solenoidal irrotational field can be expressed either as $\text{grad } \phi$ or as $\text{curl } \bar{A}$.

Taking the curl of Eq. 4.1.1 and using Eq. 4.2.1 and the vector formula

$$\text{curl}(\text{curl } \bar{a}) = \text{grad div } \bar{a} - \nabla^2 \bar{a} \quad 4.2.3$$

one obtains that \bar{A} must satisfy the Poisson equation

$$\nabla^2 \bar{A} = -\bar{\zeta} \quad 4.2.4$$

Because of its relationship with the vorticity, $\text{curl } \bar{A}$ will be called vortical velocity

$$\bar{v}_v = \text{curl } \bar{A} \quad 4.2.5$$

Eq. 4.2.1 will be rewritten as

$$\bar{v} = \text{grad } \phi + \bar{v}_v \quad 4.2.6$$

In addition to the equation for \bar{A} , an equation for ϕ and one for $\bar{\zeta}$ are required. These are derived in Sections 4.4 and 4.5 respectively. Before doing that, however, an extension of Bernoulli's theorem is needed. In potential barotropic flows the equation for the potential is obtained by replacing Dp/Dt in the continuity equation with its expression obtained from Bernoulli's theorem. For viscous rotational flows such a theorem does not exist. Therefore a generalization of Bernoulli's theorem is presented in Section 4.3.

4.3. Generalized Bernoullian Theorem

The main innovation introduced in this work is the introduction of a generalized Bernoullian theorem. There exist several so-called Bernoullian theorems (Serrin, p. 153). The classical one is for unsteady irrotational inviscid barotropic (i.e., $p=p(\rho)$) flow in a conservative field (i.e., $f=-\text{grad } \Omega$, where Ω is the potential energy).

$$H = \frac{\partial \phi}{\partial t} + \frac{1}{2} \bar{v} \cdot \bar{v} + \int \frac{dp}{\rho} + \Omega = H(t) \quad 4.3.1$$

Less known is the one for steady but rotational inviscid barotropic flow (Serrin, p. 153)

$$H = \frac{1}{2} \bar{v} \cdot \bar{v} + \int \frac{dp}{\rho(p)} + \Omega = \text{constant over a Lamb surface} \quad 4.3.2$$

(a Lamb surface is a surface defined by a network of vortex lines and streamlines).

It is apparent that Eq. 4.3.2 is considerably different from the classical Bernoulli's theorem, Eq. 4.3.1, in that H is not constant in the whole field, but is a function of the location. Such a function is constant on the Lamb surfaces, or if one prefers, $\text{grad } H$ is normal to the Lamb surfaces. The objective of this subsection is to obtain an expression similar to Eq. 4.3.2 for viscous compressible flows, i.e., an expression which reduces to Eq. 4.3.2 for inviscid, barotropic steady flows. It is assumed in the rest of this work that \bar{f} is conservative so that

$$\bar{f} = -\text{grad } \Omega \quad 4.3.3$$

In order to accomplish this, consider Cauchy's equation of motion, Eq. 4.1.2, which, using Eq. 4.1.4 and 4.3.3, may be rewritten as

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} + \frac{1}{2} \text{grad } \bar{v}^2 - \bar{v} \times \bar{\zeta} = \\ -\text{grad } \Omega - \text{grad } h + T \text{grad } S + \frac{1}{\rho} \text{div } \bar{V} \end{aligned} \quad 4.3.4$$

or, using Eq. 4.2.6,

$$\begin{aligned} \text{grad} \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} \bar{v} \cdot \bar{v} + h + \Omega \right) = \\ - \frac{\partial \bar{v}_v}{\partial t} + \bar{v} \times \bar{\zeta} + T \text{grad } S + \frac{1}{\rho} \text{div } \bar{V} \end{aligned} \quad 4.3.5$$

It is apparent that the right-hand side of Eq. 4.3.5 is irrotational (for instance by taking the curl of both sides). Hence the integral of minus the (dissipative) terms on the right hand side,

$$f_D(\bar{x}) = \int_{\bar{x}} \left(\frac{\partial \bar{v}_v}{\partial t} - \bar{v} \times \bar{\zeta} - T \text{grad } S - \frac{1}{\rho} \text{div } \bar{V} \right) \cdot d\bar{x} \quad 4.3.6$$

is path independent (suitable branch surfaces are introduced for multiply connected regions), i.e., f_D is only a function of \bar{x} and is such that

$$\text{grad } f_D = \frac{\partial \bar{v}_v}{\partial t} - \bar{v} \times \bar{\zeta} - T \text{grad } S - \frac{1}{\rho} \text{div } \bar{V} \quad 4.3.7$$

Combining Eqs. 4.3.5 and 4.3.7 yields

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \bar{v} \cdot \bar{v} + h + \Omega + f_D = H_0(t) \quad 4.3.8$$

where Π_0 is a function of time (for instance, if the fluid at $x_1 = -\infty$ is in uniform translation with speed U_∞ , $\Pi_0 = U_\infty^2 / 2 + h_\infty + f_{D\infty}$ where h_∞ and $f_{D\infty}$ are the values of h and f_D at $x_1 = -\infty$). Equation 4.3.8 may be written as:

$$H = \frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + h + \Omega = \text{constant (over } f_D\text{-surface)} \quad 4.3.9$$

(where a f_D -surface is a surface defined by $f_D = \text{constant}$).

Equation 4.39 is the generalized Bernoullian theorem. It is apparent that it reduces to Eq. 4.3.2 for steady, isentropic ($S=\text{constant}$), inviscid ($\bar{V} = 0$) flows, since, in this case,

$$H = H_0 - f_D \quad 4.3.10$$

is constant on the Lamb surfaces (for, $\bar{v} \times \bar{\zeta}$ is by definition normal to Lamb surfaces).

It is understood that Eq. 4.3.8 is a 'formal' (rather than a substantial) generalization of Bernoulli's theorem. In other words, no claim is made here about the discovery of a new physical concept. More appropriately, Eq. 4.3.6 should be thought of as a convenient formal expression that can be used to obtain a differential equation for the scalar potential (see Section 4.4). The determination of the function $f_D(\bar{x})$ is a problem as complicated as the original one, unless the hypotheses under which Eq. 4.3.2 (isentropic inviscid steady flow) is valid are satisfied. However, it will be seen in Section 4.4 that, at least for steady state, the evaluation of f_D is not necessary (see Eq. 4.4.10). Therefore, at least in that sense, the above generalization is a powerful tool because it allows for the derivation of the differential equation for the scalar potential ϕ . Also, it may be worth noting that, integrating along a streamline.

$$f_D(\bar{x}) = \int_{\bar{x}} \left(\frac{\partial \bar{v}}{\partial t} - T \text{ grad } S - \frac{1}{\rho} \text{div } \bar{V} \right) \cdot d\bar{x} \quad 4.3.11$$

4.4. Differential Equation for Potential

Consider the continuity equation, Eq. 4.1.1, which may be rewritten as

$$\text{div } \bar{v} = - \frac{1}{\rho} \frac{D\rho}{Dt} \quad 4.4.1$$

Combining with Eqs. 4.2.1 and 4.2.2 yields

$$\nabla^2 \phi = - \frac{1}{\rho} \frac{D\rho}{Dt} \quad 4.4.2$$

Assuming h and S as fundamental variables of state, one obtains

$$\nabla^2 \phi = - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial h} \right)_S \frac{Dh}{Dt} + \frac{\partial \rho}{\partial S} \Big|_h \frac{DS}{Dt} \quad 4.4.3$$

Note that (see Eq. 4.C.3)

$$\left. \frac{1}{\rho} \frac{\partial \rho}{\partial h} \right|_S = \left. \frac{\partial \rho}{\partial p} \right|_S = \frac{1}{a^2} \quad 4.4.4$$

where a is the usual isentropic speed of sound.

Introducing for notational convenience

$$B = - \left. \frac{1}{\rho} \frac{\partial \rho}{\partial S} \right|_h \quad 4.4.5$$

Eq. 4.4.3 may be rewritten as

$$\nabla^2 \phi = - \frac{1}{a^2} \frac{Dh}{Dt} + B \frac{DS}{Dt} \quad 4.4.6$$

The enthalpy h can be obtained from the generalized Bernoullian theorem, Eq. 4.3.8, to yield

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{a^2} \frac{D}{Dt} \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} + f_D + \Omega \right) + B \frac{DS}{Dt} \\ &= \frac{1}{a^2} \frac{D}{Dt} \left(\frac{\partial \phi}{\partial t} + \bar{\mathbf{v}}_c \cdot \bar{\mathbf{v}} + f_D + \Omega \right) + B \frac{DS}{Dt} \end{aligned} \quad 4.4.7$$

(where c indicates that $\bar{\mathbf{v}}$ is kept constant while applying the operator D/Dt), or

$$\nabla^2 \phi = \frac{1}{a^2} \left(\frac{D_c^2 \phi}{Dt^2} + \bar{\mathbf{v}} \cdot \frac{D \bar{\mathbf{v}}_c}{Dt} + \frac{D f_D}{Dt} + \frac{D \Omega}{Dt} \right) + B \frac{DS}{Dt} \quad 4.4.8$$

where in D_c^2 / Dt^2 the term $\bar{\mathbf{v}}$ in the first substantial derivative is kept constant during the second material differentiation.

Note that, according to Eq. 4.3.7

$$\frac{D f_D}{Dt} = \frac{\partial f_D}{\partial t} + \bar{\mathbf{v}} \cdot \text{grad } f_D = \frac{\partial f_D}{\partial t} + \bar{\mathbf{v}} \cdot \frac{\partial \bar{\mathbf{v}}_c}{\partial t} - T \bar{\mathbf{v}} \cdot \text{grad } S - \frac{1}{\rho} \bar{\mathbf{v}} \cdot \text{div } \bar{\mathbf{V}} \quad 4.4.9$$

which indicates that, as mentioned above, in steady state the explicit evaluation of f_D is not necessary.

4.5. Vorticity Dynamics Equation

In order to complete the formulation an equation for $\bar{\zeta}$ is needed. Taking the curl of Eq. 4.3.4 yields

$$\text{curl} \frac{\partial \bar{\mathbf{v}}}{\partial t} - \text{curl}(\bar{\mathbf{v}} \times \bar{\zeta}) = \text{grad } T \times \text{grad } S + \text{curl} \left(\frac{1}{\rho} \text{div } \bar{\mathbf{V}} \right) \quad 4.5.1$$

Noting that

$$\text{curl}(\bar{\mathbf{a}} \times \bar{\mathbf{b}}) = -\bar{\mathbf{a}} \cdot \text{grad } \bar{\mathbf{b}} + \bar{\mathbf{b}} \cdot \text{grad } \bar{\mathbf{a}} + \bar{\mathbf{a}} \text{div } \bar{\mathbf{b}} - \bar{\mathbf{b}} \text{div } \bar{\mathbf{a}} \quad 4.5.2$$

one obtains

$$\begin{aligned}\text{curl}(\bar{v} \times \bar{\zeta}) &= -\bar{v} \cdot \text{grad} \bar{\zeta} + \bar{\zeta} \cdot \text{grad} \bar{v} + \bar{v} \text{div} \bar{\zeta} - \bar{\zeta} \text{div} \bar{v} \\ &= -\bar{v} \cdot \text{grad} \bar{\zeta} + \bar{\zeta} \cdot \text{grad} \bar{v} + \frac{1}{\rho} \frac{D\rho}{Dt} \bar{\zeta}\end{aligned}\quad 4.5.3$$

and, combining with Eq. 4.5.1 and noting that

$$\text{curl} \frac{\partial \bar{v}}{\partial t} = \frac{\partial}{\partial t} \text{curl} \bar{v} = \frac{\partial \bar{\zeta}}{\partial t} \quad 4.5.4$$

one obtains

$$\begin{aligned}\frac{\partial \bar{\zeta}}{\partial t} + \bar{v} \cdot \text{grad} \bar{\zeta} - \bar{\zeta} \cdot \text{grad} \bar{v} - \frac{1}{\rho} \frac{D\rho}{Dt} \bar{\zeta} \\ = \text{grad} T \times \text{grad} S + \text{curl} \left(\frac{1}{\rho} \text{div} \bar{V} \right)\end{aligned}\quad 4.5.5$$

or

$$\frac{D}{Dt} \left(\frac{\bar{\zeta}}{\rho} \right) = \frac{\bar{\zeta}}{\rho} \cdot \text{grad} \bar{v} + \frac{1}{\rho} \text{grad} T \times \text{grad} S + \frac{1}{\rho} \text{curl} \left(\frac{1}{\rho} \text{div} \bar{V} \right) \quad 4.5.6$$

which is an extension to unsteady viscous fluid of Vazsonyi vorticity dynamics equation (Serrin²⁷, p. 189).

4.6. Summary of results

The results can be summarized as follows. The fundamental unknowns are the potential ϕ , the vorticity $\bar{\zeta}$, and the entropy S . The corresponding equations are:

Potential (Eq. 4.4.8)

$$\nabla^2 \phi = \frac{1}{a^2} \left(\frac{D_c \phi}{Dt^2} + \bar{v} \cdot \frac{D\bar{v}_v}{Dt} + \frac{Df_D}{Dt} + \frac{D\Omega}{Dt} \right) + \frac{DS}{Dt} \quad 4.6.1$$

Vorticity (Eq. 4.5.6)

$$\frac{D}{Dt} \left(\frac{\bar{\zeta}}{\rho} \right) = \frac{\bar{\zeta}}{\rho} \cdot \text{grad} \bar{v} + \frac{1}{\rho} \text{grad} T \times \text{grad} S + \frac{1}{\rho} \text{curl} \left(\frac{1}{\rho} \text{div} \bar{V} \right) \quad 4.6.2$$

Entropy (Eq. 4.1.3)

$$\rho T \frac{DS}{Dt} = \Phi - \text{div} \bar{q} \quad 4.6.3$$

In addition the enthalpy is given by the generalized Bernoullian theorem (Eq. 4.3.8)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \bar{v} \cdot \bar{v} + h + \Omega + f_D = H_0 \quad 4.6.4$$

whereas the velocity is given by Eq. 4.2.1

$$\bar{v} = \text{grad } \phi + \text{curl } \bar{A} \quad 4.6.5$$

where \bar{A} satisfies Eq. 4.2.4

$$\nabla^2 \bar{A} = -\bar{\zeta} \quad 4.6.6$$

The above equations have more unknowns than equations and require equations of state giving ρ, p and T as functions of h and S (such as those for an ideal gas considered in Section 4.A) and constitutive relations for the heat flux vector, \bar{q} and the viscous stress tensor, \bar{V} , such as the Fourier law for heat conduction and the Cauchy-Poisson law of viscosity (which yields the well known Navier-Stokes equations of motion) which are discussed in Section 4.D.

The above equations with appropriate state equations and constitutive equations may be solved approximately as discussed in Section 5 for both attached and separated flows. The possible advantages of the scalar/vector potential formulation over the primitive - variable solution of Navier-Stokes equation are also discussed in Section 5.

It may be noted that for inviscid ($\bar{V} = 0$), adiabatic ($\bar{q} = 0$) flows, in the absence of external forces ($\bar{\Omega} = 0$), the above equations reduce to

$$\nabla^2 \phi = \frac{1}{a^2} \left(\frac{D_t^2 \phi}{Dt^2} + \frac{Df_D}{Dt} + \bar{v} \cdot \frac{D\bar{v}}{Dt} \right) \quad 4.6.7$$

$$\frac{D}{Dt} \left(\frac{\bar{\zeta}}{\rho} \right) = \frac{\bar{\zeta}}{\rho} \cdot \text{grad } \bar{v} + \frac{1}{\rho} \text{grad } T \times \text{grad } S \quad 4.6.8$$

$$\frac{DS}{Dt} = 0 \quad 4.6.9$$

Note that integrating Eq. 4.3.6 along a streamline (using $d\bar{x} = (\bar{v}/|\bar{v}|) ds = \bar{v} d\theta$, where s is the arclength along a streamline and $d\theta = ds/|\bar{v}|$)

$$f_D = \int \left(\frac{\partial \bar{v}}{\partial t} \cdot \bar{v} + T \frac{\partial S}{\partial t} \right) d\theta \quad 4.6.10$$

since

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \bar{v} \cdot \text{grad } S = 0 \quad 4.6.11$$

In particular, for steady state, noting that (see Eq. 4.3.7) $\bar{v} \cdot \text{grad } f_D = 0$,

$$\nabla^2 \phi = \frac{1}{a^2} \bar{v} \cdot \text{grad } \frac{v^2}{2} \quad 4.6.12$$

$$\bar{v} \cdot \text{grad} \left(\frac{\bar{\zeta}}{\rho} \right) = \frac{\bar{\zeta}}{\rho} \cdot \text{grad } \bar{v} + \frac{1}{\rho} \text{grad } T \times \text{grad } S \quad 4.6.13$$

$$\frac{1}{2} \bar{v} \cdot \bar{v} + h = \text{constant on Lamb surfaces} \quad 4.6.14$$

$$S = \text{constant along streamlines} \quad 4.6.15$$

4.A. Equation of State for an Ideal Gas

In this section the necessary equations of state for an ideal gas are presented. An ideal gas is a gas which satisfies the equation

$$\frac{p}{\rho} = p\tau = RT \quad 4.A.1$$

where R is a constant.

This implies (see Eq. 3.A.6)

$$\left. \frac{\partial e}{\partial \tau} \right|_T = 0 \quad 4.A.2$$

and therefore

$$e = e(T) = \int^T c_v(T) dT \quad 4.A.3$$

where

$$c_v = \frac{de}{dT} \quad 4.A.4$$

In addition (see Eq. 3.6.4 and 4.A.1)

$$h = h(T) = \int^T c_p(T) dT \quad 4.A.5$$

where

$$c_p = c_v + R \quad 4.A.6$$

Finally (see Eq. 3.2.4)

$$dS = \frac{de}{T} + \frac{p}{T} d\tau \quad 4.A.7$$

or

$$S - S_0 = \int^T c_v \frac{dT}{T} + R \ln \tau \quad 4.A.8$$

Equation 4.A.5 can be solved for T to yield $T = T(h)$. This can be used in Eq. 4.A.8 to yield $\tau = \tau(S, h)$ and finally Eq. 4.A.1 can be used to obtain $p = \rho RT = p(S, h)$ which are the three desired equations of state relating temperature, density and pressure to the fundamental state variables S and h .

In particular, for ideal gases with constant coefficients, i.e.,

$$c_v = \text{constant} \quad 4.A.9$$

and hence (see Eq. 4.A.6)

$$c_p = \text{constant} \quad 4.A.10$$

Eqs. 4.A.3 and 4.A.5 yield

$$e = c_v T \quad 4.A.11$$

$$h = c_p T \quad 4.A.12$$

(where the constant of integration has been avoided by choosing* $e = h = 0$ for $T = 0$), whereas Eq. 4.A.7 yields

$$dS = c_v \frac{dT}{T} + R \frac{d\tau}{\tau} = c_v \frac{dp}{p} + c_p \frac{d\tau}{\tau} \quad 4.A.13$$

or

$$p\tau^\gamma = e^{S/c_v} \quad 4.A.14$$

(where the constant of integration has been avoided by choosing* $S = 0$ for $p = \tau = 1$) and

$$RT \tau^{\gamma-1} = e^{S/c_v} \quad 4.A.15$$

Hence, one obtains explicit expressions

$$T = T(h) = h/c_p \quad 4.A.16$$

$$\begin{aligned} \tau = \tau(h, S) &= (R h e^{-S/c_v} / c_p)^{1/(\gamma-1)} = \\ &= \left(\frac{\gamma-1}{\gamma} h \right)^{1/(\gamma-1)} e^{S/R} \end{aligned} \quad 4.A.17$$

$$p = p(h, S) = \left(h \frac{\gamma-1}{\gamma} \right)^{\gamma/(\gamma-1)} e^{-S/R} \quad 4.A.18$$

It may be worth noting that (see Eq. 4.4.5)

$$B = - \frac{1}{p} \frac{\partial p}{\partial S} \Big|_h = \frac{1}{\tau} \frac{\partial \tau}{\partial S} \Big|_h = \frac{1}{R} \quad 4.A.19$$

is a constant. Also, note that the fundamental equation of state, Eq. 3.2.1, is (see Eq. 4.A.12)

$$e = \frac{1}{\gamma-1} \frac{1}{\tau^{\gamma-1}} e^{S/c_v} \quad 4.A.20$$

i.e., the formulation for ideal gases with constant coefficients is equivalent to postulating the above equation for the fundamental equation of state.

* This is consistent with the third law of thermodynamics.

4.E. Constitutive Relations

In this section, the constitutive relations are discussed: in particular it is indicated how the Cauchy-Poisson law of viscosity (see Eq. 3.3.4)

$$\bar{V} = \lambda \Theta \bar{I} + 2\mu \bar{D} \quad 4.E.1$$

(where $\Theta = \text{div } \bar{V}$, and the viscosity coefficients λ and μ are positive functions of thermodynamic variables) and the Fourier law of heat conduction

$$\bar{q} = -k \text{ grad } T \quad 4.E.2$$

(where the conductivity coefficient k is a positive function of the thermodynamic variables) may be obtained by introducing certain additional restrictive (but 'reasonable') postulates.

The first postulate introduced here is that the entropy condition, Eq. 3.4.4, be satisfied by each individual term (see Eq. 3.3.6)

$$\bar{\Phi} = \bar{V} : \bar{D} > 0 \quad 4.E.3$$

and

$$\bar{q} \cdot \text{grad } T > 0 \quad 4.E.4$$

These are the mathematical expressions for the familiar statements that mechanical deformation dissipates energy (transforming it into heat), and that the heat flows in the direction of the temperature gradient.

Next, consider a set of four postulates introduced by Stokes (see Serrin²⁷, p. 231)

1. \bar{V} is a continuous function of the deformation tensor \bar{D} and is independent of all other kinematic quantities.
2. \bar{V} does not depend explicitly on the position \bar{x} (spacial homogeneity).
3. There is no preferred direction in space (isotropy).
4. When $\bar{D} = 0$, $\bar{V} = 0$ as well.

A fluid satisfying the above postulates is called Stokesian. Mathematically speaking, the first, second and fourth postulates imply

$$\bar{V} = f(\bar{D}) \quad (f(0)=0) \quad 4.E.5$$

(The dependence upon two thermodynamic variables, such as p and T , while not stated explicitly, is not being excluded). The third postulate implies that Eq. 4.E.3 is invariant under all

rectangular coordinate transformations.

In addition, consider another postulate:

5. The function $f(\bar{D})$ is linear in \bar{V} .

These postulates are sufficient to obtain the Cauchy-Poisson law of viscosity, Eq. 4.B.1. (A proof of this equation is given in Serrin²⁷ pp. 233-234). It may be worth noting that if the fifth postulate, which is the most restrictive one, is removed then, as shown in Serrin pp. 231-232, $f(\bar{D})$ must be of the type

$$\bar{T} = \alpha \bar{I} + \beta \bar{D} + \gamma \bar{D}^2 \quad 4.B.6$$

where α , β and γ are functions of the invariants of \bar{D} (as well as of the thermodynamic variables).

Introducing similar postulates for \bar{q} yields the Fourier law for heat conduction, Eq. 4.E.2.

4.C. Speed of Sound

Note that

$$d\rho = \left. \frac{\partial \rho}{\partial h} \right|_S dh + \left. \frac{\partial \rho}{\partial S} \right|_h dS \quad 4.C.1$$

On the other hand, using Eq. 3.6.2

$$\begin{aligned} d\rho &= \left. \frac{\partial \rho}{\partial p} \right|_S dp + \left. \frac{\partial \rho}{\partial S} \right|_p dS = \left. \frac{\partial \rho}{\partial p} \right|_S (\rho dh - \rho T dS) + \left. \frac{\partial \rho}{\partial S} \right|_p dS \\ &= \rho \left. \frac{\partial \rho}{\partial p} \right|_S dh + \left(- \left. \frac{\partial \rho}{\partial p} \right|_S \rho T + \left. \frac{\partial \rho}{\partial S} \right|_p \right) dS \end{aligned} \quad 4.C.2$$

Comparing Eqs. 4.C.1 and 4.C.2 yields

$$\frac{1}{\rho} \left. \frac{\partial \rho}{\partial h} \right|_S = \left. \frac{\partial \rho}{\partial p} \right|_S = \frac{1}{a^2} \quad 4.C.3$$

where a is the usual (isentropic) speed of sound, defined as

$$a = \left(\left. \frac{\partial p}{\partial \rho} \right|_S \right)^{1/2} \quad 4.C.4$$

Section 5

Concluding Remarks

5.1. Comments

A general formulation for viscous, compressible flows has been presented. In order to discuss the advantages of the formulation, consider the case of the attached flow around an isolated wing at a high Reynolds number. In this case the region of nonzero vorticity is limited to a thin region around the wing (boundary layer) and a region behind the wing (wake), and the numerical results for the pressure distribution obtained under the assumption of isentropic irrotational flow are generally in excellent agreement with the experimental ones, both in the subsonic and supersonic regimes. This would indicate that the effects of the presence of the vorticity in the field are small and therefore the solution to Navier-Stokes equations may be obtained as a correction of the potential flow formulation. In order to better understand this, consider the relationship between the inviscid adiabatic formulation (Eqs. 4.6.7 to 4.6.11) and the potential formulation. If the flow is initially isentropic (e.g., at rest at $t = 0$), Eq. 4.6.9 yields that $S = S_0$ (where S_0 is a constant) at all times. Then if the flow is initially irrotational (e.g., at rest at $t = 0$), Eq. 4.6.8 yields that $\bar{\zeta} = 0$ at all times. Therefore $\bar{v}_v = 0$ at all times, and Eq. 4.6.10 yields $f_D = 0$ at all times, and Eq. 4.6.7 reduces to the classical formulation for the velocity potential.

Next consider the relationship between inviscid adiabatic formulation (Eqs. 4.6.7 to 4.6.11) and the formulation for viscous conductive flows (Eqs. 4.6.1 to 4.6.6). In both cases we will assume that the flow is initially at rest. If the viscosity and conductivity coefficients are small (i.e., high Reynolds number flows), then Eq. 4.6.3 yields that S is approximately constant in the outer region (the fluid volume minus the boundary layer and the wake region). In addition Eq. 4.6.2 yields that the vorticity is approximately equal to zero in the outer region. Next consider the velocity, \bar{v}_v , induced by the vorticity (see Eq. 4.6.6). With a suitable choice for the boundary conditions for Eq. 4.2.4, it is possible to obtain that $\bar{A} = 0$ (approximately) in the outer region, since the velocity obtained from Eq. 4.6.6 is irrotational in the outer region. As a consequence of the above remarks, $\partial S / \partial t$ and $\partial \bar{v}_v / \partial t$ are zero in the outer region and so is f_D . Therefore, the additional (nonpotential) terms in Eq. 4.6.1 are all approximately equal to zero in the outer region and hence the only modification in the equation for the velocity potential due to the effect of viscosity is the presence of some apparent sources of mass in the region of the boundary layer and the wake. In this case an integral equation formulation would be ideal to approach this problem (at least in the subsonic and supersonic regimes).

5.2. Recommendation for future work

A general formulation for scalar/vector-potential decomposition for viscous compressible flows has been presented. This formulation should not be considered the last word on the subject but merely the first. Several issues have not been addressed here, such as separated flows, turbulence, and the presence of shock waves in the field and the vorticity generated by them. However before addressing such issues it seems appropriate to assess the formulation presented here by developing a numerical algorithm (and corresponding computer program) and comparing the results against existing numerical and experimental ones.

Section 6

References

1. Orzag, S.A. and Israeli, M., 'Numerical Simulation of Viscous Incompressible Flows', Annual Review of Fluid Mechanics, Vol. 6, 1974, pp. 281-318.
2. MacCormack, R.W., and Lomax, H., 'Numerical Solution of Compressible Viscous Flows', Annual Review of Fluid Mechanics, Vol. 11, 1979, pp. 289-316.
3. Shen, S., 'Finite-Element Methods in Fluid Mechanics', Annual Review of Fluid Mechanics, Vol. 9, 1977, pp. 173-206.
4. Lighthill, M.J., 'On Displacement Thickness,' Journal of Fluid Mechanics, Vol. 4, 1958, pp. 383-392.
5. Lighthill, M.J., 'Introduction: Boundary Layer Theory', Laminar Boundary Layers, J. Rosenhead, ed., Oxford University Press, Oxford (1963), pp. 54-61.
6. Lamb, H., Hydrodynamics, Sixth Edition, Cambridge University Press, 1932.
7. Stokes, G., 'On the Dynamical Theory of Diffraction', Camb. Trans. ix. 1849.
8. Helmholtz, 'Ueber Integrale der hydrodynamischen Gleichungen Welche den Wirbelbewegulgen entsprechen', Crelle, lv. 1858.
9. Richardson, S.M., and Cornish, A.R.H., 'Solution of Three-Dimensional Incompressible Flow Problems', Journal of Fluid Mechanics, Vol. 82, 1977, pp. 309-319.
10. Bykhovskiy, E.B., and Smirnov, N.V., 'On Orthogonal Expansions of the Space of Vector Functions which are Square-Summable, Over a Given Domain and the Vector Analysis Operators', Trudy Math. Inst. Steklova, Vol. 59, No. 5, 1960.
11. Hirasaki, G.J., and Hellums, J.D., 'A General Formulation of the Boundary Conditions on the Vector Potential in Three-Dimensional Hydrodynamics', Quart. Appl. Math., Vol. 26, No. 3, April 1968, pp. 331-342.
12. Hiraski, G.J., and Hellums, J.D., 'Boundary Conditions on The Vector and Scalar Potentials in Viscous Three-Dimensional Hydrodynamics', Quart. Appl. Math., Vol. 28, No. 2, July 1970, pp. 293-296.
13. Ladyzhenskaya, O.A., The Mathematical Theory of Viscous Incompressible Flows, Gordon and Breach, N.Y., 1963.

14. Payne, R.B., 'Calculations of Unsteady Viscous Flow Past a Circular Cylinder', Journal of Fluid Mechanics, Vol. 4, part 1, May 1958, pp. 81-86.
15. Thompson, J.F., Shanks, S.P., and Wu, J.C., 'Numerical Solution of Three-Dimensional Navier Stokes Equations Showing Trailing Tip Vortices', AIAA Journal, Vol. 12, No. 6, 1974.
16. Kinney, R.B., and Paolino, M.A., 'Flow Transient Near the Leading Edge of a Flat Plate Moving Through a Viscous Fluid', Journal of Applied Mechanics, Vol. 41, No. 4, 1974.
17. Panniker, P., and Lavan, Z., 'Flow Past Impulsively Started Bodies Using Green's Functions', Journal of Comp. Phys. Vol. 18, 1975.
18. Wu, J.C., 'Numerical Boundary Conditions for Viscous Flow Problems', AIAA Journal, Vol. 14, No. 8, August 1976.
19. Kinney, R.B., 'Two-Dimensional Flow Past an Airfoil in an Unsteady Airstream', AGARD-CP-227, Unsteady Aerodynamics, Ottawa, 1977.
20. Mehta, U.B., 'Dynamic Stall of an Oscillating Airfoil', AGARD-CP-227, Unsteady Aerodynamics, Ottawa, 1977.
21. Wu, J.C., Sampath, C., and Sankar, N.L., 'A Numerical Study of Unsteady Viscous Flows Around Airfoils', AGARD-CP-227, Unsteady Aerodynamics, Ottawa, 1977.
22. Aregbesola, Y.A.S., and Burley, D.M., 'The Vector and Scalar Potential Method for the Numerical Solution of Two- and Three-dimensional Navier-Stokes Equations', Journal of Computational Physics, Vol. 24, 1977, pp. 398-415.
23. Sugavanam, A., and Wu, J.C., 'Numerical Study of Separated Turbulent Flow Over Airfoils', AIAA Paper 80-1441, AIAA 13th Fluid and Plasma Dynamics Conference, 1980.
24. Morino, L., 'A Finite Element Method for Rotational Incompressible Aerodynamics', Boston University, College of Engineering, TN-74-04, December 1974.
25. Morino, L., 'Nonpotential Aerodynamics for Windmills in Shear-Winds', ERDA/NSF 2975 Wind Energy Conversion Systems Workshop, Washington, D.C., June 9-11, 1975.

26. Udelson, D., Kantelis, J., and Morino, L., 'Potential-Vorticity Method for the Analysis of Wind Effects on Buildings and Structures', Boston University, College of Engineering, TR-81-01, 1981.
27. Serrin, J., 'Mathematical Principles of Classical Fluid Mechanics' in Encyclopedia of Physics Ed. S. Fluegge, Vol. VIII/1, Fluid Dynamics I, Springer Verlag, 1959.

1. Report No. NASA CR-3921		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle Scalar/Vector Potential Formulation for Compressible Viscous Unsteady Flows				5. Report Date August 1985	
				6. Performing Organization Code	
7. Author(s) Luigi Morino				8. Performing Organization Report No. TR-84-01	
9. Performing Organization Name and Address Institute for Computer Application Research and Utilization in Science, Inc. P.O. Box 194 Lincoln, MA 01773				10. Work Unit No.	
				11. Contract or Grant No. NAS1-17317	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546				13. Type of Report and Period Covered Contractor Report	
				14. Sponsoring Agency Code 505-33-43-12	
15. Supplementary Notes Final Report Langley Technical Monitor: E. Carson Yates, Jr.					
16. Abstract A scalar/vector potential formulation for unsteady viscous compressible flows is presented. The scalar/vector potential formulation is based on the classical Helmholtz decomposition of any vector field into the sum of an irrotational and a solenoidal field. The formulation is derived from fundamental principles of mechanics and thermodynamics. The governing equations for the scalar potential and vector potential are obtained, without restrictive assumptions on either the equation of state or the constitutive relations for the stress tensor and the heat flux vector.					
17. Key Words (Suggested by Author(s)) Unsteady Flow Viscous Flow Compressible Flow Unsteady Viscous Compressible Flow				18. Distribution Statement Unclassified-Unlimited Subject Category 02	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 46	22. Price A03		

National Aeronautics and
Space Administration

Washington, D.C.
20546

Official Business

Penalty for Private Use, \$300

BULK RATE
POSTAGE & FEES PAID
NASA Washington, DC
Permit No. G-27



POSTMASTER: If Undeliverable (Section 158
Postal Manual) Do Not Return
